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On the closability of paranormal operators

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ABSTRACT

We show two examples of operators acting on some Hilbert space and having invariant domains: a paranormal operator, which is not closable and a paranormal and closable operator, which closure is not paranormal. We start by establishing some general lemmas and propositions associating the families of operators mentioned above.

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0. Introduction

The notion of a paranormal operator dates back to 1960s and is due to V. Istrătescu – in [4] he named them “operators of class N ”. T. Furuta in [2] introduced the term “paranormal operators”. This class can be seen as a generalization of other important classes: hyponormal operators (i.e. T satisfying $T^*T - TT^* \geq 0$) and subnormal and normal operators (as every (sub)normal operator is hyponormal). In subsequent years paranormal operators have been the subject of further research: for example we know [3] that a paranormal operator T is compact if and only if T^n is compact for some $n \in \mathbb{N}$. Moreover, compact paranormal operator is normal and if T is paranormal (and invertible), then T^{-1} also is [5]. We know [2] that if T is paranormal, then for every $n \geq 2$, T^n is also paranormal. Many topological properties of paranormal operators have been studied in [7]. There appeared also a number of papers on connection between paranormality and spectra of operators, e.g. we know [6] that every paranormal operator has its norm equal to spectral radius and that a paranormal operator which spectrum is contained in the unit circle is always unitary [5]. In [1] the link between paranormal, normaloid and essentially normal operators has been shown. Paranormality appears also to be an important property when studying various problems in operator theory (see e.g. [8]).

It is therefore quite surprising that the questions of the closability and the properties of the closure of the paranormal operators still have been opened. For subnormal as well as for hyponormal operators it is known that they are closable, and moreover, the closure of the subnormal (hyponormal) operator is also subnormal (hyponormal), respectively. These results allow to reduce the studies of these classes of operators to closed operators only. So the question whether analogous results hold for paranormal operators appears quite natural and important.

In this paper we fill this gap and give negative answer to both questions. We provide two examples of operators acting on some Hilbert space: a paranormal operator, which is not closable and next a paranormal and closable operator, which closure is not paranormal; both operators have invariant domains.

1. Basic definitions

Fix \mathbb{K} as one of the fields: \mathbb{R} or \mathbb{C} . All spaces appearing in the paper will be considered over the field \mathbb{K} .

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Here and subsequently $\{e_i\}_{i=1}^{\infty}$ denotes a canonical basis in a Hilbert space l^2 of square-summable sequences. \mathbb{K}^n denotes here the n -dimensional space embedded into l^2 , i.e.:

$$\mathbb{K}^n = \text{span}(e_1, e_2, \dots, e_n).$$

S^{n-1} is a unit sphere in \mathbb{K}^n , i.e.

$$S^{n-1} = \{x \in \mathbb{K}^n : \|x\| = 1\}.$$

c_{00} is the space of finitely nonzero sequences, we can write it as

$$c_{00} = \bigcup_{n=1}^{\infty} \mathbb{K}^n.$$

Definition 1.1. A linear operator $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$, where \mathcal{H} is a Hilbert space, is *paranormal* if

$$\|Ax\|^2 \leq \|x\| \|A^2x\|, \quad x \in D(A^2).$$

Remark 1.2. Some authors use the following definition:

A bounded linear operator A on a Hilbert space \mathcal{H} is called *paranormal* if $\|A^2x\| \geq \|Ax\|^2$ for every unit vector $x \in D(A^2)$,

which is obviously equivalent to Definition 1.1.

Definition 1.3. A linear operator $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is *closable* if the closure of its graph in $\mathcal{H} \oplus \mathcal{H}$ is the graph of some operator (called the closure of A).

Remark 1.4. A linear operator $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is closable if and only if for every sequence (x_n) in $D(A)$, tending to zero, it holds

$$\lim_{n \rightarrow \infty} Ax_n = y \implies y = 0.$$

We also introduce a Hilbert space

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} l^2 = \left\{ (x_1, x_2, \dots) : x_i \in l^2, \sum_{i=1}^{\infty} \|x_i\|^2 < \infty \right\}$$

and its dense subspace

$$\mathcal{D} = \bigoplus_{n=1}^{\infty} c_{00} = \{ (x_1, x_2, \dots) : x_i \in c_{00} \text{ and } x_j = 0 \text{ for almost every } j \}.$$

Definition 1.5. We will call a linear operator $A : c_{00} \rightarrow c_{00}$

- (i) *diagonal* if $Ae_n = a_n e_n$ for all n , where $a_n \in \mathbb{K} \setminus \{0\}$,
- (ii) *subdiagonal* if $Ae_n = a_{n1}e_1 + \dots + a_{nn}e_n$ for all n and for some $a_{n1}, \dots, a_{nn} \in \mathbb{K}$, $a_{nn} \neq 0$.

Remark 1.6.

- (i) An operator A is subdiagonal if and only if $A\mathbb{K}^n = \mathbb{K}^n$ for all n ,
- (ii) a diagonal operator is closable.

2. Paranormal, closable, diagonal and subdiagonal operators – lemmas and propositions

In this section we prove a few lemmas and propositions which will provide the tools for our main examples.

Proposition 2.1. Let $A_1, A_2, \dots : c_{00} \rightarrow c_{00}$ be linear operators. Define $\mathcal{A} : \mathcal{D} \rightarrow \mathcal{D}$ by:

$$\mathcal{A}(x_1, x_2, \dots) = (0, A_1x_1, A_2x_2, \dots).$$

Then

- (i) \mathcal{A} is closable in \mathcal{H} if and only if each A_j is closable in l^2 ,

(ii) if for all $j \in \mathbb{N}$ and for all $x \in c_{00}$

$$\|A_j x\|^2 \leq \|A_{j+1} A_j x\| \|x\|, \quad (2.1)$$

then \mathcal{A} is paranormal.

Proof.

(i) The proof is immediate by definition.

(ii) For $x = (x_1, x_2, \dots) \in \mathcal{D}$ we have $\mathcal{A}^2 x = (0, 0, A_2 A_1 x_1, A_3 A_2 x_2, \dots)$. Hence

$$\|\mathcal{A}^2 x\|^2 = \sum_{j=1}^{\infty} \|A_{j+1} A_j x_j\|^2.$$

Also

$$\begin{aligned} \|\mathcal{A} x\|^2 &= \sum_{j=1}^{\infty} \|A_j x_j\|^2 \leq \sum_{j=1}^{\infty} \|A_{j+1} A_j x_j\| \|x_j\| \\ &\leq \left(\sum_{j=1}^{\infty} \|A_{j+1} A_j x_j\|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} \|x_j\|^2 \right)^{1/2} = \|\mathcal{A}^2 x\| \|x\| \end{aligned}$$

(the first inequality follows by the assumption, the second is the Schwarz inequality) and (ii) is proved. \square

Lemma 2.2. Let $A : c_{00} \rightarrow c_{00}$ be a subdiagonal operator. Then there exists a diagonal operator $B : c_{00} \rightarrow c_{00}$, such that for all $x \in c_{00}$

$$\|Ax\|^2 \leq \|BAx\| \|x\|. \quad (2.2)$$

Proof. Choose any real sequence (c_n) such that $0 < c_1 < c_2 < \dots < 1$.

We will show existence of an operator B , satisfying condition more restrictive than needed, namely such that

$$\|Ax\|^2 \leq c_n \|BAx\|, \quad x \in S^{n-1} \quad (2.3)$$

for all n .

B is diagonal, hence it is determined by a sequence (β_n) , $\beta_n \neq 0$, such that $Be_n = \beta_n e_n$.

We now construct this sequence by induction.

A is subdiagonal, which implies that $Ae_n = \alpha_n e_n + f_n$ for some $\alpha_n \neq 0$, $f_n \in \mathbb{K}^{n-1}$ ($f_1 = 0$).

Moreover, for all $n \in \mathbb{N}$ $A|_{\mathbb{K}^{n-1}}$ is invertible, hence $f_n = Ag_n$ for some $g_n \in \mathbb{K}^{n-1}$.

For $n = 1$ it suffices to set $\beta_1 = \frac{|\alpha_1|}{c_1}$.

Assume that (2.3) holds for some $n \geq 1$. We will find β_{n+1} such that (2.3) holds for $n + 1$. Let $x \in S^n$. Then $x = px' + qe_{n+1}$ for some $x' \in S^{n-1}$ and $p, q \in \mathbb{K}$ such that $|p|^2 + |q|^2 = 1$. Hence

$$\begin{aligned} Ax &= pAx' + qAe_{n+1} = pAx' + qf_{n+1} + q\alpha_{n+1}e_{n+1} \\ &= A(px' + qg_{n+1}) + q\alpha_{n+1}e_{n+1} = Ax'' + q\alpha_{n+1}e_{n+1}, \end{aligned}$$

where $x'' = px' + qg_{n+1} \in \mathbb{K}^n$.

Set

$$m_n := \inf \left\{ \|Ay\| : y \in \mathbb{K}^n, \|y\| \geq \frac{1}{2} \right\}$$

and

$$M_{n+1} := \sup \{ \|Ay\|^2 : y \in S^n \}.$$

Note that by invertibility and continuity of $A|_{\mathbb{K}^n}$ we have $m_n > 0$. Obviously M_{n+1} is also positive.

Fix $\varepsilon > 0$ satisfying inequalities:

$$(1 + \varepsilon \|g_{n+1}\|) \left(1 + \frac{|\alpha_{n+1}|^2}{m_n^2} \varepsilon^2 \right) \leq \frac{c_{n+1}}{c_n} \quad (2.4)$$

and

$$\varepsilon < \frac{1}{2} \cdot \frac{1}{1 + \|g_{n+1}\|}. \quad (2.5)$$

Let

$$\beta_{n+1} = \frac{M_{n+1}}{c_{n+1}\varepsilon|\alpha_{n+1}|}. \quad (2.6)$$

In order to show that β_{n+1} is such that (2.3) holds for $n+1$, we will consider the two following cases:

1. Let $|q| < \varepsilon$. Then $|p| = \sqrt{1 - |q|^2} > \sqrt{1 - \varepsilon^2} > 1 - \varepsilon$.

According to the above remark and by (2.5), we have

$$\begin{aligned} \|x''\| &\geq |p|\|x'\| - |q|\|g_{n+1}\| \geq (1 - \varepsilon) - \varepsilon\|g_{n+1}\| \\ &= 1 - \varepsilon(1 + \|g_{n+1}\|) \geq 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Hence

$$\|Ax''\| \geq m_n.$$

It follows that

$$\begin{aligned} \|Ax\|^2 &= \|Ax''\|^2 + |q|^2|\alpha_{n+1}|^2 \leq \|Ax''\|^2 + \frac{\|Ax''\|^2}{m_n^2}|\alpha_{n+1}|^2\varepsilon^2 \\ &= \left(1 + \frac{|\alpha_{n+1}|^2}{m_n^2}\varepsilon^2\right)\|Ax''\|^2. \end{aligned}$$

On the other hand

$$\|x''\| \leq |p|\|x'\| + |q|\|g_{n+1}\| \leq 1 + \varepsilon\|g_{n+1}\|.$$

Moreover

$$BAx = BAx'' + q\alpha_{n+1}\beta_{n+1}e_{n+1},$$

hence, due to orthogonality of the summands, $\|BAx\| \geq \|BAx''\|$. From the induction hypothesis

$$\|Ax''\|^2 \leq c_n\|BAx''\|\|x''\| \leq c_n(1 + \varepsilon\|g_{n+1}\|)\|BAx\|.$$

Hence finally

$$\|Ax\|^2 \leq c_n(1 + \varepsilon\|g_{n+1}\|)\left(1 + \frac{|\alpha_{n+1}|^2}{m_n^2}\varepsilon^2\right)\|BAx\|.$$

By (2.4) we get $\|Ax\|^2 \leq c_{n+1}\|BAx\|$.

2. Let $|q| \geq \varepsilon$. Then we have

$$\|BAx\| \geq |q|\alpha_{n+1}|\beta_{n+1}| \geq \varepsilon|\alpha_{n+1}|\beta_{n+1}|,$$

hence by definition of M_{n+1} and (2.6) we obtain

$$\|Ax\|^2 \leq M_{n+1} \leq c_{n+1}\|BAx\|.$$

This completes the proof. \square

Lemma 2.3. Let $A_1, A_2 : c_{00} \rightarrow c_{00}$ be linear operators, satisfying

$$\|A_1x\|^2 \leq \|A_2A_1x\|\|x\| \quad \text{for all } x \in c_{00}$$

and let A_2 be subdiagonal. Then there exist diagonal operators $A_3, A_4, \dots : c_{00} \rightarrow c_{00}$ such that a linear operator $\mathcal{A} : \mathcal{D} \rightarrow \mathcal{D}$ given by

$$\mathcal{A}(x_1, x_2, \dots) = (0, A_1x_1, A_2x_2, \dots)$$

is paranormal. Furthermore, \mathcal{A} is closable if and only if A_1, A_2 are closable.

Proof. The existence of A_3, A_4, \dots satisfying (2.2) follows from Lemma 2.2 if we put consecutively $A = A_n$, $B = A_{n+1}$ for $n = 2, 3, \dots$ (note that every diagonal operator is also subdiagonal).

The paranormality of \mathcal{A} follows from Proposition 2.1(ii).

The criterion of closability follows from Proposition 2.1(i) and Remark 1.6(ii) (we only need to know whether A_1 and A_2 are closable, because A_3, A_4, \dots , as diagonal, are). \square

3. Main examples

Below in Examples 3.2 and 3.4 we construct two operators: a paranormal, yet not closable, and a paranormal and closable one, which closure is not paranormal.

Lemma 3.1. Let $A_1, A_2 : c_{00} \rightarrow c_{00}$ be linear operators, defined on a canonical basis:

$$A_1 e_1 = 0, \quad A_1 e_n = e_n, \quad n \geq 2,$$

$$A_2 e_n = e_1 + e_n, \quad n \geq 1.$$

Then A_1, A_2 satisfy assumptions of Lemma 2.3, i.e. A_2 is subdiagonal and for all $x \in c_{00}$, $\|A_1 x\|^2 \leq \|A_2 A_1 x\| \|x\|$.

Proof. Obviously A_2 is subdiagonal.

For any $x_1, \dots, x_n \in \mathbb{K}$ and $x = x_1 e_1 + \dots + x_n e_n$ we have

$$A_1 x = x_2 e_2 + \dots + x_n e_n,$$

$$A_2 A_1 x = (x_2 + \dots + x_n) e_1 + x_2 e_2 + \dots + x_n e_n = A_1 x + (x_2 + \dots + x_n) e_1$$

and terms $A_1 x$ and $(x_2 + \dots + x_n) e_1$ are orthogonal, hence

$$\|A_2 A_1 x\| \geq \|A_1 x\|$$

and

$$\|A_1 x\|^2 \leq \|A_1 x\| \|x\| \leq \|A_2 A_1 x\| \|x\|. \quad \square$$

Example 3.2. Let A_1, A_2 be linear operators defined in Lemma 3.1 and let \mathcal{A} be the linear operator, considered in Lemma 2.3. Then \mathcal{A} is paranormal, has invariant domain, but is not closable.

Proof. The paranormality of \mathcal{A} follows from Lemmas 3.1 and 2.3.

Moreover A_2 is not closable (which, according to Lemma 2.3, shows that \mathcal{A} is not closable either), because taking

$$x_n = \frac{1}{n}(e_1 + \dots + e_n)$$

we get $x_n \xrightarrow{n \rightarrow \infty} 0$, while $A_2 x_n = e_1 + x_n \xrightarrow{n \rightarrow \infty} e_1 \neq 0$. \square

Lemma 3.3. Let $A_1, A_2 : c_{00} \rightarrow c_{00}$ be linear operators, defined on a canonical basis:

$$A_1 e_n = e_{2n}, \quad n \geq 1,$$

$$A_2 e_1 = e_1,$$

$$A_2 e_{2n} = n e_{2n}, \quad A_2 e_{2n+1} = e_{2n+1} - (e_2 + e_4 + \dots + e_{2n}), \quad n \geq 1.$$

Then A_1, A_2 satisfy assumptions of Lemma 2.3, i.e. A_2 is subdiagonal and for all $x \in c_{00}$, $\|A_1 x\|^2 \leq \|A_2 A_1 x\| \|x\|$.

Proof. Obviously A_2 is subdiagonal.

For any $x_1, \dots, x_n \in \mathbb{K}$ and $x = x_1 e_1 + \dots + x_n e_n$ we have

$$\|A_1 x\| = \|x\|, \quad \|A_2 A_1 x\|^2 = \sum_{k=1}^n |k x_k|^2 \geq \sum_{k=1}^n |x_k|^2 = \|x\|^2,$$

hence

$$\|A_1 x\|^2 = \|x\| \|x\| \leq \|A_2 A_1 x\| \|x\|. \quad \square$$

Example 3.4. Let A_1, A_2 be linear operators defined in Lemma 3.3 and let \mathcal{A} be the linear operator, considered in Lemma 2.3. Then \mathcal{A} is paranormal and closable, has invariant domain, but $\overline{\mathcal{A}}$ is not paranormal.

Proof. The proof will be divided into 3 steps:

(i) The paranormality of \mathcal{A} .

It follows from Lemmas 3.3 and 2.3.

(ii) The closability of A_1 and A_2 .

The closability of A_1 is obvious (since A_1 is an isometry).

To show the closability of A_2 we take $x \in c_{00}$ of the form

$$x = \sum_{n \geq 1} u_n e_{2n} + \sum_{n \geq 0} w_n e_{2n+1}.$$

We have

$$\begin{aligned} A_2 x &= \sum_{n \geq 1} n u_n e_{2n} + \sum_{n \geq 0} w_n e_{2n+1} - \sum_{n \geq 1} w_n \sum_{1 \leq k \leq n} e_{2k} \\ &= \sum_{n \geq 1} n u_n e_{2n} + \sum_{n \geq 0} w_n e_{2n+1} - \sum_{k \geq 1} \left(\sum_{n \geq k} w_n \right) e_{2k} \\ &= \sum_{n \geq 1} \left(n u_n - \sum_{k \geq n} w_k \right) e_{2n} + \sum_{n \geq 0} w_n e_{2n+1}. \end{aligned}$$

Hence

$$\langle A_2 x, e_{2n+1} \rangle = w_n = \langle x, e_{2n+1} \rangle, \quad \langle A_2 x, e_{2n} \rangle = n u_n - \sum_{k \geq n} w_k$$

and

$$\begin{aligned} \langle A_2 x, e_{2n+2} - e_{2n} \rangle &= (n+1) u_{n+1} - n u_n + w_n \\ &= \langle x, e_{2n+2} \rangle + n \langle x, e_{2n+2} - e_{2n} \rangle + \langle x, e_{2n+1} \rangle. \end{aligned} \quad (3.1)$$

Take $x^{(m)} \xrightarrow{m \rightarrow \infty} 0$ such that $A_2 x^{(m)} \xrightarrow{m \rightarrow \infty} y$ for some $y \in l^2$.

Then for each $n \in \mathbb{N}$

$$\lim_{m \rightarrow \infty} \langle x^{(m)}, e_{2n+1} \rangle = 0, \quad \lim_{m \rightarrow \infty} \langle A_2 x^{(m)}, e_{2n+1} \rangle = \langle y, e_{2n+1} \rangle,$$

while for all $n \in \mathbb{N}$ we have $\langle A_2 x^{(m)}, e_{2n+1} \rangle = \langle x^{(m)}, e_{2n+1} \rangle$, which yields $\langle y, e_{2n+1} \rangle = 0$. Similarly, by (3.1), we have

$$\begin{aligned} \langle y, e_{2n+2} - e_{2n} \rangle &= \lim_{m \rightarrow \infty} \langle A^2 x^{(m)}, e_{2n+2} - e_{2n} \rangle \\ &= \lim_{m \rightarrow \infty} \left(\langle x^{(m)}, e_{2n+2} \rangle + n \langle x^{(m)}, e_{2n+2} - e_{2n} \rangle + \langle x^{(m)}, e_{2n+1} \rangle \right) = 0, \end{aligned}$$

and so $\langle y, e_{2n+2} \rangle = \langle y, e_{2n} \rangle$ for all n . This, in view of the square summability of the Fourier coefficients of y , yields $\langle y, e_{2n} \rangle = 0$ for all n . As a consequence, we get $y = 0$, which proves the closability of A_2 .

(iii) The non-paranormality of \bar{A} .

Let

$$x_n = \sum_{k=1}^n \frac{1}{k} e_k, \quad y_n = \sum_{k=1}^n \frac{1}{k} e_{2k}.$$

Obviously $x_n \in c_{00}$ and $y_n = A_1 x_n$. Moreover

$$\begin{aligned} x_n &\xrightarrow{n \rightarrow \infty} x = \sum_{k=1}^{\infty} \frac{1}{k} e_k \in l^2, \\ y_n &\xrightarrow{n \rightarrow \infty} y = \sum_{k=1}^{\infty} \frac{1}{k} e_{2k} \in l^2, \end{aligned}$$

hence $\bar{A}_1 x = y$.

Set $y'_n = y_n + e_{2n+1}$. Then

$$\begin{aligned} A_2 y'_n &= A_2 y_n + A_2 e_{2n+1} = \sum_{k=1}^n \frac{1}{k} A_2 e_{2k} + e_{2n+1} - \sum_{k=1}^n e_{2k} \\ &= \sum_{k=1}^n e_{2k} + e_{2n+1} - \sum_{k=1}^n e_{2k} = e_{2n+1}. \end{aligned}$$

Set

$$y_n'' = \frac{1}{n}(y_1' + \cdots + y_n') = \frac{1}{n} \sum_{k=1}^n y_k + \frac{1}{n} \sum_{k=1}^n e_{2k+1}.$$

Since $\lim_{k \rightarrow \infty} y_k = y$, we also have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n y_k = y.$$

Because

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e_{2k+1} = 0,$$

we get also $\lim_{n \rightarrow \infty} y_n'' = y$. On the other hand

$$A_2 y_n'' = \frac{1}{n} \sum_{k=1}^n e_{2k+1} \xrightarrow{n \rightarrow \infty} 0,$$

hence $y \in D(\overline{A_2})$ and $\overline{A_2}y = 0$.

For $x, y \in \mathcal{D}$ such that $x = (x, 0, 0, \dots)$, $y = (0, y, 0, 0, \dots)$ we have $x \in D(\overline{A^2})$, $\overline{A}x = y$, $\overline{A}y = 0$ and, as a consequence, $\overline{A^2}x = 0$. Therefore inequality

$$\|\overline{A}x\|^2 \leq \|\overline{A^2}x\| \|x\|$$

cannot be satisfied, unless $y = \overline{A}x = 0$, which does not hold.

This contradicts the paranormality of \overline{A} . \square

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